Traveled Distance Minimization and Hierarchical Strategies for Robotic Networks

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***Abstract*—We study the distance optimal assignment of n mobile robots to an equal number of targets under communication and target-sensing constraints. Extending previous results over uniform distributions, we show that when the robots and targets assume the same but arbitrary distribution over the unit square, a carefully designed distributed hierarchical strategy has expected travel distance that matches the best known upper bound assuming global communication and infinite target-sensing range. In a sense, our result shows that for target assignment problems in robotic networks, local optimality also offers good guarantees on global optimality.**

***Index Terms*—Robotic networks, optimization, network connectivity, hierarchical strategies.**

I. Introduction

We study permutation-invariant assignments of a set of networked robots to a set of targets of equal cardinality, with a primary focus on minimizing the total path distance. Both robots and targets are assumed to be uniformly randomly distributed in a two-dimensional unit square. In an earlier work, under communication and target-sensing limitations, we provided optimality guarantees, in terms of necessary and sufficient conditions, as well as asymptotically optimal or suboptimal strategies when the conditions for optimality cannot be satisfied. In this paper, we extend the result from uniform distribution to arbitrary distribution. In particular, we show that the bound on distance optimality remains essentially the same as the uniform distribution case.

The problem of target assignment in robotic networks requires solving an *assignment* (or *matching*) problem. Assignment problems are extensively studied in the area of combinatorial optimization, with efficient algorithms available for solving many of its variations [1], [4], [5], [7], [10], [16], [33]. If we instead put more emphasis on multi-robot systems, the problems of robotic task allocation [14], [25], [26], [32], swarm reconfiguration [8], multi-robot path planning [15], [22], [27], and multi-agent consensus [9], [13], [17], [18] come up. For a review on some of these topics, see [6].

Among many related work, the closest one is perhaps [23], in which the performance of several classes of algorithms for achieving time optimality (i.e., minimizing the time until every target is occupied) were established. In contrast, we focus on minimizing the total distance traveled by all robots. The total distance serves as a proper proxy to quantities such as the energy consumption of all robots. Simple examples show that a distance-optimal solution for the target assignment problem generally does not imply time optimality and vice versa [31].

Our work is also related to the study on the connectivity of wireless networks. If n robots are uniformly randomly distributed in a unit square, then each robot needs to have $k = \Theta(\log n)$ nearest neighbors for the entire network to be asymptotically connected [29]. In particular, the authors of [29] showed that $k < 0.074 \log n$ leads to an asymptotically disconnected network whereas $k > 5.1774 \log n$ guarantees asymptotic connectivity. This pair of bounds was subsequently improved [3]. These nearest neighbor based connectivity models were further studied in [11], [12], [19], to list a few. In these work, a *geometric graph* structure is often used [21].

The rest of the paper is organized as follows. In Section II, we introduce notations and formally define the problem that we study. Sections III briefly review results from our previous work [30]. Section IV then develops results using the newly introduced region-based communication model for arbitrary robot and target distributions. We conduct simulations in Section V to confirm our theoretical findings and conclude in Section VI.

II. Problem Statement

Notations. The symbols $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$ denote the set of real numbers, the set of positive reals, and the set of positive integers, respectively. We use $\log(\cdot)$ to denotes the natural logarithm function; $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$are the ceiling and floor functions. $|\cdot|$ denotes the cardinality for sets and the absolute values for real numbers. $\|\cdot\|_2$ denotes the Euclidean 2-norm function. A unit square Q is defined as the set $[0,1] \times [0,1] \subset \mathbb{R}^2$. $E(\cdot)$ denotes a probabilistic event and the probability with which an event e occurs is denoted as $\mathbf{P}(e)$. The expectation of a random variable X is denoted as $\mathbf{E}[X]$. Given two functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = O(g(x))$ (resp. $\Omega(g(x))$) if $\lim_{x\to\infty} f(x)/g(x) < \infty$ (resp. $\lim_{x\to\infty} f(x)/g(x) \geq c > 0$). Note here that "=" behaves as "∈". If $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$, then we say $f(x) = \Theta(g(x))$. Finally, $f(x) = o(g(x))$ (respectively, $f(x) = \omega(g(x))$) if and only if $f(x) = O(g(x))$ (respectively, $f(x) = \Omega(g(x))$) and $\neg(f(x) = \Theta(x))$.

Target assignment in robotic networks. Let $X^0 = \{x_1^0, \ldots, x_n^0\}, Y^0 = \{y_1^0, \ldots, y_n^0\} \subset Q$ be two uniformly randomly selected point sets of cardinality n. The superscript emphasizes that these points are obtained at the start time $t = 0$. Place n point robots, labeled $a_1, \ldots, a_n$, on the points in X^0, with robot a_i occupying x_i^0. In general, we denote robot a_i's location (coordinates) at time $t \geq 0$ as $x_i(t)$. The basic task, to be

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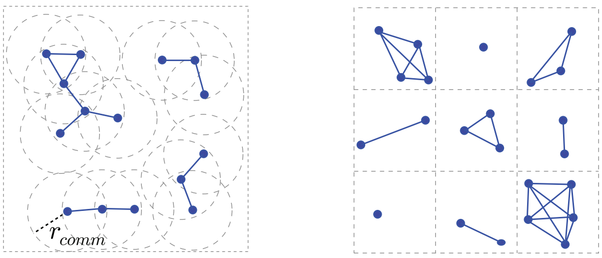
formally defined, is to move the robots so that at some *final time* $t^f \geq 0$, every $y \in Y^0$ is occupied by a robot (we may assume that there is a final time t_i^f for each robot a_i, such that $x_i(t) \equiv x_i(t_i^f)$ for $t \geq t_i^f$). For convenience, we also refer to X^0 and Y^0 as the set of initial locations and the set of target locations, respectively.

Motion model: The control space for a robot a_i is $\dot{x}_i = u_i$ with $\|u_i\|_2 \in \{0, 1\}$. We assume that robots' sizes are negligible with respect to the distance they travel and ignore collisions between robots.

Communication Model 1: We use two communication models in this paper. In the first communication model, a robot a_i may communicate with other robots within a disc of radius r_{comm} centered at $x_i(t)$. At any given time $t \geq 0$, we define the (undirected) *communication graph* $G(t)$ as follows, which is a geometric graph that changes over time. $G(t)$ has n vertices $v_1, \ldots, v_n$, corresponding to robots $a_1, \ldots, a_n$, respectively. There is an edge between two vertices v_i and v_j if the corresponding robot locations $x_i(t)$ and $x_j(t)$, respectively, satisfy $\|x_i(t) - x_j(t)\|_2 \leq r_{\text{comm}}$. Figure 1(a) provides an example of a (disconnected) communication graph.

Given our focus on distance optimality, we make the simplifying assumption that all robots corresponding to vertices in a connected component of the communication graph may exchange information as needed instantaneously. In other words, robots in a connected component of $G(t)$ can be effectively treated as a single robot insofar as decision making is concerned.



(a) Comm. model 1 (b) Comm. model 2

Fig. 1. (a) The communication graph (solid blue nodes and edges) for a given set of robots under Communication Model 1 with a communication radius of r_{comm}. Robots (blue dots) in the same component can freely communicate. (b) The communication graph for a set of robots under Communication Model 2 with $m = b^2 = 9$.

Communication Model 2: In the second communication model, the unit square Q is divided into some $m = b^2$ equal sized smaller squares (regions). Robots within each region can communicate with one another but robots from different regions cannot exchange information (see, e.g., Fig. 1(b)). This model mimics the natural (geometrical) resource allocation process in which supplies and demands are first matched locally; the surpluses and deficits within the each region then get balanced out at larger regions, giving rise to a hierarchical decision process.

Target-sensing model: We assume that a robot is aware of a point $y \in Y^0$ if $\|y - x_i(t)\|_2 \leq r_{\text{sense}}$, the *target-sensing radius.*

The problem we consider in this paper is defined as follows.

Problem 1 (Target Assignment in Robotic Networks) *Given X^0, Y^0, r_{sense}, and Communication Model 1 or 2 with r_{comm}, find a control strategy $\mathbf{u} = [u_1, \ldots, u_n]$, such that for some $0 \leq t_i^f < \infty$ and some permutation σ of the numbers $1, \ldots, n$, $x_i(t_i^f) = y^0_{\sigma(i)}$ for all $1 \leq i \leq n$.*

Over all feasible solutions to an instance of Problem 1, we are interested in minimizing the total distance traveled by all robots, which can be expressed as

$$D_n = \sum_{i=1}^{n} \int_0^{t_i^f} \|\dot{x}_i(t)\|_2 dt. \quad (1)$$

As an accurate proxy to measures such as the energy consumption of the entire system, the cost defined in (1) is an appropriate objective in practice. Unless otherwise specified, *distance optimality* refers to minimizing D_n. Assuming that robots must follow continuous paths, we let D_n^* denote the best possible D_n, which may or may not be achievable depending on the capabilities of the robots (e.g, if the robots cannot follow straight-line paths, then $D_n > D_n^*$). Let $\mathscr{U}$ denote the set of all possible control strategies that solve Problem 1 given a fixed set of capabilities for the robots, $\inf_{\mathscr{U}} D_n$ is then the greatest lower bound achievable under these capabilities.

III. Non-Asymptotic Bounds for Probabilistic Optimality Guarantees

In this section, we review some of our recent results first appeared in [30] under Communication Model 1. Intuitively, global availability of information (*i.e.*, X^0, Y^0) is required to guarantee optimality (i.e., $\inf_{\mathscr{U}} D_n = D_n^*$), because global assignment is otherwise impossible in general at $t = 0$. For example, as $r_{\text{sense}} \to 0$, the robots must search for the targets before assignments can be made; it is unlikely that the paths taken by the robots toward the targets will be straight lines, which is required to obtain D_n^*. This notion is formalized in the following theorem.

Theorem 1 ([30]) *In a unit square, under sensing and communication constraints* (i.e., $r_{\text{comm}}, r_{\text{sense}} < \sqrt{2}$), *$\inf_{\mathscr{U}} D_n = D_n^*$ if and only if $G(0)$ is connected and every target $y \in Y^0$ is within a distance of r_{sense} to some $x \in X^0$.*

Theorem 1 says that to ensure optimality, one may increase the number of robots, or increase one or both of r_{comm} and r_{sense}. The target assignment problem can then be solved with a centralized strategy (Strategy 1). Note that given the assignment permutation σ, each robot a_i can easily compute its straight-line path between x_i^0 and $y^0_{\sigma(i)}$. Since every robot can carry out the computation in Strategy 1, to resolve conflicting decisions and avoid unnecessary computation, we may let the highest labeled robot (e.g., a_n) dictate the assignment process. An optimal assignment in the unit square can be computed in $O(n^3)$ using the strongly polynomial Hungarian algorithm [10], [16] or other asymptotically faster algorithms [1], [28].

Strategy 1: CENTRALIZED ASSIGNMENT

Initial condition: X^0, Y^0
Outcome: permutation σ that assigns robot a_i to $y^0_{\sigma(i)}$

compute $d_{i,j} = \|x_i - y_j\|_2$ between each pair of (x_i, y_j) in which $x_i \in X^0$ and $y_j \in Y^0$
based on $\{d_{i,j}\}$, compute an optimal assignment for the robots that minimizes D_n
communicate the assignment to all robots

Given the assumption that X^0 and Y^0 follow the uniform distribution, one can expect that there will be some critical n above which the conditions given in Theorem 1 will be probabilistically satisfied. This bound can be computed using existing results on random geometric graphs [20]. In the following theorem, random geometric graphs are equivalent to our definition of communication graphs at $t = 0$.

Theorem 2 (Random Geometric Graphs [20]) *Let G be the random geometric graph obtained following the uniform distribution over the unit square for some n and r_{comm}. Then for any real number c, as $n \to \infty$ ($r_{\text{comm}} \to 0$),*

$$\mathbf{P}(G \textit{ is connected} \mid \pi n r^2_{\text{comm}} - \log n \le c) = e^{-e^{-c}}. \quad (2)$$

However, Equation (2) is an implicit formula of an asymptotic nature. We instead give bounds on n as an explicit function of r_{comm} and r_{sense}, without the asymptotic assumption.

Theorem 3 ([30]) *Fixing $0 < \varepsilon < 1$, the communication graph is connected and every target $y \in Y^0$ is observable by some robot at $t = 0$ with probability at least $1 - \varepsilon$ when*

$$n \ge \begin{cases} \lceil \frac{\sqrt{2}}{r_{\text{sense}}} \rceil^2 \log(\frac{1}{\varepsilon} \lceil \frac{\sqrt{2}}{r_{\text{sense}}} \rceil^2), & r_{\text{sense}} < \frac{\sqrt{10} r_{\text{comm}}}{5} \\ \lceil \frac{\sqrt{5}}{r_{\text{comm}}} \rceil^2 \log(\frac{1}{\varepsilon} \lceil \frac{\sqrt{5}}{r_{\text{comm}}} \rceil^2), & r_{\text{sense}} \ge \frac{\sqrt{10} r_{\text{comm}}}{5} \end{cases} \quad (3)$$

Remark. Theorem 3 says that the conditions from Theorem 1 can be guaranteed with arbitrarily high probability (via setting ε to be arbitrarily small in Theorem 3). Note that the result does not depend on n being large and therefore is non-asymptotic in nature.

We also make the observation that, by Theorem 3, for any arbitrary but fixed probability requirement (*i.e.*, by fixing ε), $n = O(\lceil \sqrt{2}/r_{\text{sense}} \rceil^2 \log \lceil \sqrt{2}/r_{\text{sense}} \rceil)$ or $n = O(\lceil \sqrt{5}/r_{\text{comm}} \rceil^2 \log \lceil \sqrt{5}/r_{\text{comm}} \rceil)$ robots will ensure the conditions from Theorem 1 are met. Interestingly, for high probability guarantees, these many robots are also necessary, as detailed in the following theorem.

Theorem 4 ([30]) *For uniformly randomly distributed robots in a unit square with a communication radius r_{comm},*

$$n = \Theta(\frac{1}{r^2_{\text{comm}}} \log \frac{1}{r_{\text{comm}}}) \quad (4)$$

robots are necessary and sufficient to ensure a connected communication graph at $t = 0$ with arbitrarily high probability as $r_{\text{comm}} \to 0$.

Similarly, for uniformly randomly distributed targets in a unit square, with a target-sensing radius of r_{sense},

$$n = \Theta(\frac{1}{r^2_{\text{sense}}} \log \frac{1}{r_{\text{sense}}}) \quad (5)$$

robots are necessary and sufficient to ensure that every target is within r_{sense} distance of some robot at $t = 0$ with arbitrarily high probability as $r_{\text{sense}} \to 0$.

IV. ASYMPTOTIC NEAR-OPTIMALITY OF HIERARCHICAL STRATEGIES

We now shift our attention to the (region-based) Communication Model 2 and assume that $r_{\text{sense}} \ge \sqrt{2}$ (that is, every robot is aware of the entire Y^0). As discussed in [30], the assumption of $r_{\text{sense}} \ge \sqrt{2}$ does not affect asymptotic optimality. As we will see, Communication Model 2 leads to the conclusion that optimal local behavior gives rise to near-optimal global behavior for the problem we are looking at.

To set up a hierarchical structure, let $h \ge 1$ be the number of hierarchies and $m_i, 1 \le i \le h$, be the number of equal sized regions at hierarchy i, we make the following assumptions (mainly used in Theorem 5): 1. $m_1 \equiv 1$, 2. $m_{i+1} \ge m_i$, and 3. a region at a higher numbered hierarchy is contained in a single region at a lower numbered hierarchy. As an example, dividing Q into 4^{i-1} squares at hierarchy i satisfies these requirements. We call the associated strategy under these assumptions the *hierarchical divide-and-conquer* strategy, the details of which are described in Strategy 2. Note that for each region in Strategy 2, the robots can again let the highest labeled robot within the region carry out the strategy locally.

Strategy 2: HIERARCHICAL-DIVIDE-AND-CONQUER

Initial condition: $X^0, Y^0, h, m_1, \ldots, m_h$
Outcome: permutation σ that assigns robot a_i to $y^0_{\sigma(i)}$

for *each hierarchy i in decreasing order* **do**
  for *each region j, $1 \le j \le m_i$* **do**
    let n_a and n_g be the number of unmatched robots and targets in region j, respectively
    if $n_a > n_g > 0$ **then**
      pick the first n_g robots from the n_a unmatched robots and run an assignment algorithm to match them with the n_g unmatched targets in region j
    else if $n_g > n_a > 0$ **then**
      pick the first n_a targets from the n_g unmatched targets and run an assignment algorithm to match the n_a unmatched robots with these targets in region j
    else
      continue

It is clear that Strategy 2 is correct by construction because $|X^0| = |Y^0|$. Under uniform robot and target distribution, Communication Model 2, and assuming $r_{\text{sense}} \geq \sqrt{2}$, we have

Theorem 5 ([30]) *Suppose that the unit square Q is divided into m_i equal sized small squares at hierarchy i with a total of $h \geq 2$ hierarchies. For all applicable $i \geq 1$, assume that $m_{i+1} \geq m_i$ and any small square at hierarchy $i+1$ falls within a single square at hierarchy i. Then*

$$\mathbf{E}[D_n] \leq C\sqrt{n\log n} + \sum_{i=1}^{h-1}\sqrt{\frac{nm_{i+1}}{m_i}}. \quad (6)$$

Theorem 5 allows us to upper bound the performances of different hierarchical strategies depending on the choices of h and $\{m_i\}$. We observe that for fixed h and $\{m_i\}$ independent of n, the first term $C\sqrt{n\log n}$ dominates the other terms in (6) as $n \to \infty$. This implies that Strategy 2 yields assignments whose total distance is at most a constant multiple of the optimal distance. This observation is summarized in Corollary 6.

Corollary 6 *For fixed h and $m_1, \ldots, m_h$ that do not depend on n, as $n \to \infty$, in expectation, Strategy 2 yields target assignment with $D_n/D_n^* = O(1)$.*

For example, letting $h \geq 2$ and $m_i = 4^{i-1}$ at hierarchy i, we have

$$\mathbf{E}[D_n] \leq C\sqrt{n\log n} + \sum_{i=1}^{h-1}\sqrt{4n} = C\sqrt{n\log n} + 2(h-1)\sqrt{n}. \quad (7)$$

For any fixed h, as $n \to \infty$, $D_n/D_n^* = O(1)$ since D_n^* is lower bounded by $\Theta(\sqrt{n\log n})$ [2]. Constant approximation ratio can also be achieved when h and/or $\{m_i\}$ depend on n. For example, letting $h = 3$, $m_3 = \log^2 n$, and $m_2 = \log n$, then

$$\mathbf{E}[D_n] \leq C\sqrt{n\log n} + \sum_{i=1}^{2}\sqrt{n\log n} = (C+2)\sqrt{n\log n}. \quad (8)$$

Since hierarchical strategies need not to run centralized assignment algorithms for all robots, the computational part of these strategies can be significantly faster.

We now show that Theorem 5 in fact holds for arbitrary distributions over the unit square $[0,1]^2$. To bound D_n, at each hierarchy i, we need to know the number of robots that can be matched locally. Below, Lemma 7 provides an upper bound on this number. Note that Lemma 7 does not depend on m and n being large.

Lemma 7 *Suppose that the unit square Q is divided into m equal sized regions. Let the robots and targets assume the same arbitrary distribution over Q, then the expected number of robots that are not matched locally is no more than $\sqrt{2n(m-1)}$ in expectation.*

PROOF. Restricting to one of the m equal sized regions, say q_i, we may assume that the probability of a robot falling into q_i be $p_i \in [0,1]$. That is, for $x_j^0 < X^0$ and $y_j^0 \in Y^0$,

$$\mathbf{P}(x_j^0 \in q_i) = \mathbf{P}(y_j^0 \in q_i) = \frac{1}{p_i},$$

and

$$\mathbf{P}(x_j^0 \in q_i, y_j^0 \notin q_i) = \mathbf{P}(x_j^0 \notin q_i, y_j^0 \in q_i) = p_i(1-p_i),$$

in which the event $x_j^0 \in q_i \wedge y_j^0 \notin q_i$ represents a surplus of a robot in q_i and the event $x_j^0 \notin q_i \wedge y_j^0 \in q_i$ a deficit in q_i. Thus, we may view the experiment of picking x_j^0 and y_j^0 as a one step walk on the real line starting at the origin, with $p_i(1-p_i)$ probability of moving ± 1. The entire process of picking X^0 and Y^0 can then be treated as a random walk of n such steps.

Representing the outcome of picking a pair of (x_j^0, y_j^0) as a random variable Z_j (it is easy to see that $\mathbf{E}[Z_j^2] = 2p_i(1-p_i)$) under the random walk analogy and let $S_n = Z_1 + \ldots + Z_n$, we can compute the variance of S_n as

$$\begin{aligned}\mathbf{E}[S_n^2] &= \mathbf{E}[(Z_1 + \ldots + Z_n)^2] = \mathbf{E}[Z_1^2 + \ldots + Z_n^2] \\ &= n\mathbf{E}[Z_j^2] = 2p_i(1-p_i).\end{aligned}$$

Applying Jensen's inequality to the concave function $\sqrt{x}$ with $x = |S_n|^2 = S_n^2$, we have

$$\mathbf{E}[|S_n|] = \mathbf{E}[\sqrt{S_n^2}] \leq \sqrt{\mathbf{E}[S_n^2]} \Rightarrow \mathbf{E}[|S_n|] \leq \sqrt{2p_i(1-p_i)}.$$

That is, for square q_i, $\mathbf{E}[|S_n|] = \sqrt{2p_i(1-p_i)}$. Noting that the function $\sqrt{x(1-x)}$ is concave on $x \in [0,1]$, Jensen's inequality then gives us that $\mathbf{E}[\sqrt{x(1-x)}] \leq \sqrt{\mathbf{E}[x](1-\mathbf{E}[x])}$. Now let $x = p_i$ and using the fact that $\sum_{i=1}^m q_i = 1$, we have

$$\begin{aligned}\sum_{i=1}^{m}\sqrt{2np_i(1-p_i)} &= m\sqrt{2n}\sum_{i=1}^{m}\frac{1}{m}\sqrt{p_i(1-p_i)} \\ &\leq m\sqrt{2n}\sqrt{\sum_{i=1}^{m}\frac{p_i}{m}(1-\sum_{i=1}^{m}\frac{p_i}{m})} \\ &= m\sqrt{2n}\sqrt{\frac{1}{m}(1-\frac{1}{m})} = \sqrt{2n(m-1)}.\end{aligned}$$

□

Note that Lemma 7 does not require the divided regions to be squares. The distance traveled by the matched robots at the bottom hierarchy with m regions can also be easily bounded. For simplicity, we assume that these regions are equal sized squares.

Lemma 8 *Let n robots and n targets assume the same arbitrary distribution over Q. The total distance of matchings made at the bottom hierarchy with m equal sized square regions is no more than $C\sqrt{n\log n}$ in expectation for some positive constant C.*

PROOF. Since Q is divided into m squares, these squares all have a side length of $1/\sqrt{m}$. Let one such square be q_i with n_i robots (note that $\sum_{i=1}^m n_i = n$), by a result of Talagrand [24], in expectation, if we let these n_i robots match with only targets within q_i, the total distance incurred locally will not exceed $C\sqrt{n_i\log n_i/m}$. Here, C is some positive universal constant independent of the arbitrary distribution.

Note that it is not necessarily the case that all n_i robots will be matched locally in q_i, which does not affect the current proof. For some $1 \le i \le m$, it may be the case that no local matchings can be made because either $n_i = 0$ or there is no target in q_i. Let $m' \le m$ denote the number of these m squares in which local matchings can be made. The total distance incurred by local matchings is then upper bounded by

$$\sum_{i=1}^{m'} C\sqrt{\frac{n_i \log n_i}{m}} = C\frac{m'}{\sqrt{m}}\sum_{i=1}^{m'}\frac{1}{m'}\sqrt{n_i \log n_i}.$$

Here we assume that $m' > 0$, otherwise the local matchings have zero distance cost. Since the function $\varphi(x) = \sqrt{x \log x}$ is concave, by Jensen's inequality, $\mathbf{E}[\sqrt{x \log x}] \le \sqrt{\mathbf{E}[x]\log(\mathbf{E}[x])}$. Let $x = n_i$ and the expectation be carried out over the discrete uniform distribution with $1/m'$ probability each, we have

$$\begin{aligned} C\frac{m'}{\sqrt{m}}\sum_{i=1}^{m'}\frac{1}{m'}\sqrt{n_i\log n_i} &\le C\frac{m'}{\sqrt{m}}\sqrt{(\sum_{i=1}^{m'}\frac{n_i}{m'})\log(\sum_{i=1}^{m'}\frac{n_i}{m'})} \\ &= C\sqrt{\frac{m'}{m}}\sqrt{(\sum_{i=1}^{m'} n_i)(\log(\sum_{i=1}^{m'} n_i) - \log(m'))} \\ &\le C\sqrt{n\log n}. \end{aligned}$$

□

Combining Lemmas 7 and 8, we get a generalized version Theorem 5 with minimal change; the only difference is a multiplicative factor of two on the second summation term, *i.e.*, for arbitrary distribution over Q, Equation (6) becomes

$$\mathbf{E}[D_n] \le C\sqrt{n\log n} + \sum_{i=1}^{h-1} 2\sqrt{\frac{nm_{i+1}}{m_i}}, \qquad (9)$$

V. Simulation Studies

A. Number of Required Robots for a Connected $G(0)$

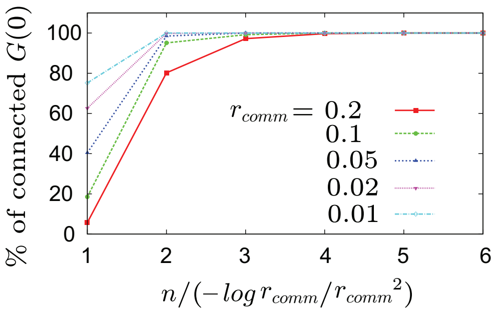


Fig. 2. Effects of n on the connectivity of $G(0)$ for different values of r_{comm}.

In this subsection, we show a result of simulation to more clearly illustrate the connectivity results from Section III. Since the bounds over r_{comm} and r_{sense} are similar, we focus on r_{comm} and confirm the requirement for the connectivity of $G(0)$ for several r_{comm}'s ranging from 0.01 to 0.2. For each fixed r_{comm}, varying numbers of robots are used starting from $n = \log(1/r_{\text{comm}})/r_{\text{comm}}^2 = -\log r_{\text{comm}}/r_{\text{comm}}^2$ (the number of robots goes as high as 3×10^5 for the case of $r_{\text{comm}} = 0.01$). 1000 trials were run for each fixed combination of r_{comm} and n; the percentages of the runs with a connected $G(0)$ were reported in the simulation result shown in Fig. 2. The simulation suggests that the bounds on n from Theorem 4 are fairly tight.

TABLE I
Comparison between Equations (2) and (3).

prob.	r_{comm} 0.2	0.1	0.05	0.02	0.01
0.1	0.001, 0.82	0.001, 0.96	0.001, 0.99	0.001, 1	0.003, 1
0.5	0.007, 0.92	0.006, 0.98	0.027, 0.99	0.064, 1	0.081, 1
0.9	0.2, 0.99	0.31, 1	0.381, 1	0.477, 1	0.502, 1
0.99	0.702, 1	0.742, 1	0.794, 1	0.834, 1	0.855, 1

To compare to the asymptotic bounds given implicitly by Equation (2), which also allows for estimation of n in terms of r_{comm} with a specified probability for obtaining a connected $G(0)$, for r_{comm} from 0.01 to 0.2, we computed n based on Equations (2) and (3) for several probabilities (from 0.1 to 0.99). We then use these n's to estimate the actual probability of having a connected $G(0)$. We list the result in Table I. Each main entry of the table has two probabilities separated by a comma, obtained using Equations (2) and (3), respectively. As we can see, (2) gives underestimates (due to its asymptotic nature) and cannot be used to provide probabilistic guarantees. On the other hand, (3) provides overestimates that guarantees the desired probability.

B. Performance of Region-Based Hierarchical Strategy

Next, we simulate Strategies 2 and evaluate its optimality, focusing on Corollary 6. Our simulation uses $r_{\text{comm}} = 0.16, 0.09, 0.057$, and 0.04, which correspond to $m = 81, 256, 625$, and 1296, respectively. The number of robots used in each simulation ranges from 100 to 10000. For each n, 10 problems are randomly generated and used across all strategies. Two hierarchical structures are used here; the first is a two-hierarchy one and the second a three-hierarchy one. The hierarchy setup and the results are given in Figures 3 and 4. We observe that these strategies generally provide very good approximations to true optimality with an approximation ratio below two across all choices of parameters.

VI. Conclusion

Focusing on the distance optimality for the target assignment problem in a robotic network setting, we characterize conditions under which optimality can be achieved and provide an explicit formula for computing the number of robots sufficient for probabilistically guaranteeing such an optimal solution. Then, we look at a region-based communication model and explicitly show that locally optimal behavior necessarily leads to near-optimal globally optimal behavior. We then further show that essentially the same optimality bound continues to hold when the robots and targets took the same but arbitrary distributions.

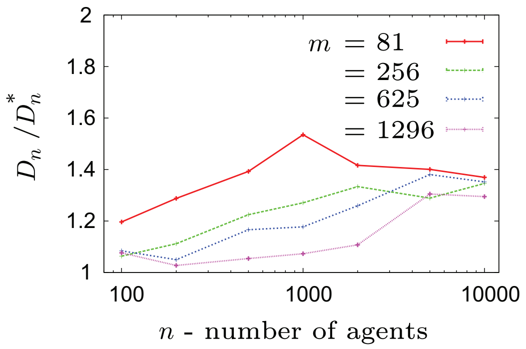


Fig. 3. The assignment cost of a two-level region-based hierarchical strategy. At the bottom hierarchy, the unit square Q is divided into $\sqrt{5}/r_{\text{comm}}$ equal-sized small squares.

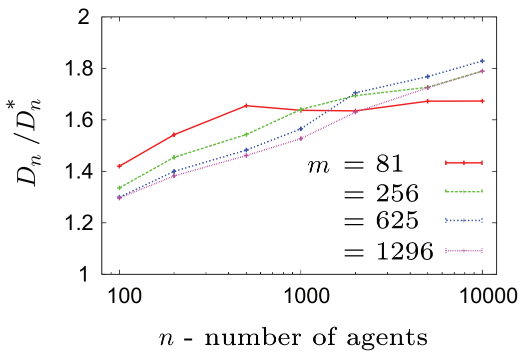


Fig. 4. The assignment cost of a three-level region-base hierarchical strategy. At the bottom hierarchy, the unit square Q is divided into $\sqrt{5}/r_{\text{comm}}$ equal-sized small squares; at the middle hierarchy, the unit square Q is divided into $\sqrt{\sqrt{5}/r_{\text{comm}}}$ equal-sized small squares

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